Isotropy, homogeneity and dipole saturation

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Abstract

A distribution of points that satisfies the property of local isotropy is not necessarily homogeneous: homogeneity is implied by the condition of local isotropy together with the *assumption of analyticity or regularity*. Here we show that the evidence of dipole saturation in galaxies (and clusters) catalogues, together with a monotone growth of the monopole, is an evidence of isotropy but not of homogeneity. This is fully compatible with a fractal structure which has the property of local isotropy, but it is non-analytic and non-homogeneous.

Subject headings: cosmology:theory - galaxy:clustering - large-scale structure of the universe.

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1 Introduction

The property of local isotropy, required by the Cosmological Principle in order to avoid propositions that imply privileged points in the Universe, together with the *assumption of analyticity*, implies homogeneity (Weinberg 1972). The Cosmological Principle is claimed to imply homogeneity directly, while homogeneity is not satisfied for a non-analytic distribution of matter, that can be locally isotropic. The fact that local isotropy does not imply homogeneity is well known (Weinberg 1972): here we discuss the property of local isotropy in the case of non-analytical distribution of matter.

Various studies (for a review see Coleman & Pietronero (1992)) have shown that the assumption of analyticity (regularity at large scales) is not supported by the experimental evidence: in fact the analysis of CfA (for galaxies) and Abell (for clusters) catalogues show highly irregular distributions of matter up to the sample limit, with fractal (multifractal if one includes mass) properties and without any clear tendency towards homogenisation.

Mandelbrot (1982) has formulated a weaker Cosmological Principle, the so-called Conditional Cosmological Principle, which does not require the complete homogeneity of space (see also Coleman & Pietronero 1992) and it is compatible with a fractal distribution of matter: this breaks the symmetry between occupied and empty points but all galaxies are *statistically equivalent* with respect to their environment (local isotropy). For a complete review of this thesis we remand to Coleman & Pietronero (1992), as well as for the objections to the criticism to this analysis (see also Ribeiro 1993). We consider here another claimed evidence for homogeneity: the dipole saturation with depth in galaxies and clusters catalogues, together with a monotone growth of the monopole.

We are going to show that this is an evidence of the isotropy of the distri-

bution but not of homogeneity. Since a fractal structure satisfies the property of local isotropy, it is fully compatible with the evidence of dipole saturation with depth, as well as an homogeneous distribution.

A number of authors have discussed the convergence of cosmological dipoles using different tracers: for example flux-weighted IRAS dipoles (Lahav et al.1988), IRAS redshift space dipole (Rowan-Robinson et al 1990), X-ray and optical clusters (Scaramella et al.1990; Plionis & Valdarnini 1991; Plionis et al 1993; Lahav et al 1989) and optical galaxies (Hudson 1993; Lahav et al.1988).

Plionis & Valdarnini (1991) using the combined the Abell-ACO clusters catalogues with $m_0 \leq 14$, found that the dipole converges at $d_{conv} \approx 150h^{-1}Mpc$, while the monopole saturates for the lacking of data for deeper depth in the catalogues, at $260h^{-1}Mpc$. Scaramella et al (1991) using a volume limited sample of Abell-ACO catalogues, found that $d_{conv} \approx 180h^{-1}Mpc$. The optical and infrared galaxies catalogues also indicate that $d_{conv} \approx 100h^{-1}Mpc$ (Rowan-Robinson et al.1990; Hudson 1993). These analysis are problematic for the lacking of data in several sky regions in the available samples, and for the accurate knowledge of the luminosity function required for the data weighting (Plionis et al 1993). In various cases however the saturation of the dipole has been interpreted as evidence for homogeneity.

We report a number of analyses on artificial distributions with a priori assigned properties, for which we have studied the behaviour of the monopole and the dipole with sample depth. Our aim is not to reproduce the actual distribution of galaxies by computer models, but we intend to show with these examples that the evidence of dipole saturation does not imply homogeneity as it is fully compatible with a non-analytic (fractal) distribution. Such a saturation identifies the scale beyond which the distribution can be considered statistically isotropic. Even in the case of a fractal distribution at small scales

with a cut-off towards homogeneity at larger scales, it is interesting to study the behaviour of the dipole and the monopole with sample depth in order to clarify the physical interpretation of an eventual dipole saturation scale.

In Sect.2 we discuss the case of an homogeneous sample and in Sect.3 we show the properties of fractal and multifractal samples. Finally we present the conclusion of this work with special emphasis to the concept of local isotropy and its implication for a fractal distribution of galaxies and clusters.

2 Homogeneous sample

We begin by considering the properties of a random distribution that is really regular and homogeneous at large scale. An homogeneous sample was generated with a random number generator distributed in a large spatial volume: also the masses are assigned by a random number generator with an uniform distribution ranging in [0,1]. Chandrasekhar (1942) provide an analytic solution for the distribution of the gravitational force in a random sample of points with random masses, in the three-dimensional euclidean space. The analytic solution is valid only in the thermodynamical limit for the number of points N and the volume of the sample V that goes to infinity, being constant the average number density $< n >$. In order to analyse the effects of a finite value of N and a finite volume V, as one has in the real catalogues, we have performed some numerical simulations to study the finite size effects and the convergency towards the asymptotic behaviour.

In our numerical simulation (Fig.1) we have obtained a result in agreement with the asymptotic one for $N = 500$ (in cube of unitary side). For such a

sample we evaluate the monopole $M(r)$ defined as:

$$M(r) = \sum_i \frac{m_i}{|\vec{r}_i|^2} \tag{1}$$

and the dipole $\vec{D}(\vec{r})$:

$$\vec{D}(\vec{r}) = \sum_i \frac{m_i \vec{r}_i}{|\vec{r}_i|^3} \tag{2}$$

Fig.2 shows that the monopole grows linearly with sample depth. The scale of dipole saturation d_{conv} (Fig.3) is quickly reached: it represents the scale beyond which the statistical isotropy of distribution is reached and the further contributions to (2) sum to zero for symmetry reason. For an homogeneous random sample the density is a constant function in space, a part from statistical fluctuations, and it is possible to compute analytically the monopole and the dipole from eqs.(1) and (2). We find that the monopole grows linearly with the sample depth as shown in the simulations beyond the scale of order of the mean Poisson separation. The dipole modulus converges as the mean density became rotationally symmetric with respect to every point: $n(\vec{r}) \approx n(|\vec{r}|)$.

3 Fractal and multifractal samples

In a fractal distribution the counting of the number of points present within a certain volume, from every occupied point, is, for a spherical volume:

$$N(R) = BR^D \tag{3}$$

where B is a constant related to the lower cut-off of the fractal, and D is the fractal dimension (for a more detailed definition of fractal dimension see Paladin & Vulpiani 1987). If $D = 3$, in the three-dimensional euclidean space, the distribution is homogeneous. The average density for a spherical sample of

radius R_s, which contains a portion of a fractal structure, is:

$$< n >= \frac{N(R_s)}{V(R_s)} = \frac{3BR_s^{D-3}}{4\pi} \tag{4}$$

From (4) it follows that in a fractal structure $< n >$ is not a well defined quantity, i.e. it depends from the sample depth ($< n >$ is constant if $D = 3$). The conditional density form an occupied point is defined as:

$$\Gamma(r) = S(r)^{-1}\frac{dN(r)}{dr} = \frac{D}{4\pi}Br^{D-3} \tag{5}$$

where $S(r)$ is the area of a spherical shell of radius r. Eqs. (3) and (5) hold from every point of the system, when considered as origin, and this means that every point of the system has the same type of environment: in other words a fractal structure satisfies the property of local isotropy. The average number density (4) is highly singular in every occupied point and this means that a fractal is non-analytic (Pietronero 1986).

The distribution of matter in a sample is described by the density function:

$$\rho(\vec{r}) = \sum_i m_i \delta(\vec{r} - \vec{r_i}) \tag{6}$$

For a fractal structure the analytic calculation of the dipole modulus and the monopole, contrary to the trivial case of the homogeneous sample, is not possible anymore, because one has to know the complete distribution of mass in space to resolve eqs. (1) and (2) explicit. So we have preceded with the analysis of artificial distributions. Fig.4 shows a simple deterministic fractal: as the structure repeats at different scales in a self-similar way it is obvious that the dipole saturates with depth quickly from every occupied point, while the monopole is an increasing function of depth. This is a particular example that shows that isotropy and homogeneity are different properties for non-analytic

distribution. In order to consider a more general case we can construct a random fractal or multifractal distribution with a random β model (Paladin & Vulpiani 1987).

We first keep the mass constant in each point and evaluate numerically the monopole and the dipole: in doing this we are only considering the fractal distribution of the support without any correlation between space and mass distribution. In order to consider also this correlation, found in the analysis of the CfA catalogue (Coleman & Pietronero 1992), we have also considered a random multifractal distribution. In this case we assign a mass to each point proportional to measure associated with the fragmentation process (Benzi et al. 1984). However there is no sensible change in the results for these models. By locating the observer in a random point of the distribution (far from the boundaries) we analyze the dipole and the monopole dependence on sample depth. In Fig.5 and Fig.6, we show the results for fractal respectively of dimension D=1.66 and D=1.85, while in Fig.7 D=1.4, equal to the fractal dimension found in the catalogues analysis (Coleman & Pietronero 1992).

We find that the monopole is an increasing function of sample depth: this is obviously due to the fact that the monopole measures the total gravitational field (as the dipole measures the gravitational force) and it increases with the number of points: going to larger depth, also in a fractal structure, we are simply including more points. The dependence of $M(\vec{r})$ is a power law:

$$M(\vec{r}) = Br^{\alpha} \tag{7}$$

with the exponent α that depends strongly from the topological properties of the realization: this is due to the fact that a fractal is characterized by having structures at all scales and also the fluctuations in the spatial distribution of these structures are present at all scales. For the same reason the saturation

depth d_{conv} of the dipole modulus depends on the particular realization of the fractal. In any case we can conclude that a fractal (and a multifractal) can be locally isotropic with a finite value of d_{conv}. The particular values of d_{conv} and α are related to morphological properties of the system that cannot be characterized simply by the value of the fractal dimension.

4 Conclusion

In the previous discussion we have stressed the fact that the property of local isotropy, required by the Cosmological Principle, without the assumption of analyticity, does not imply homogeneity. A fractal is locally isotropic, but non-analytic in every occupied point, and non-homogeneous.

We have analyzed several artificial distributions with a priori assigned properties. For the case of an homogeneous sample we have shown that the monopole grows linearly with sample depth, while the dipole saturates at the scale beyond which the isotropy is statistically reached. In the case of a fractal and multifractal distributions, generated with a random β model, the monopole follows a power low with exponent α, and the scale of dipole saturation d_{conv} is finite. Both α and d_{conv} depend strongly on the realization of the fractal because it is characterized by having structures at all scale as well as fluctuations in the spatial distribution of these structures: it is not possible to relate α and d_{conv} simply to the value of the fractal dimension as they depend from the morphological properties of the system.

In the case of cosmological dipoles observed in galaxies (and clusters) catalogues, from the analysis of the convergence depth d_{conv}, we obtain information on the scale beyond which the distribution of galaxies (clusters) have reached, statistically, the isotropy. One cannot conclude that the distribution is homo-

geneous beyond d_{conv} because the property of local isotropy does not imply homogeneity without assumption of analyticity: the fact that d_{conv} is finite is fully compatible with a fractal structure.

In summary we have shown that a distribution of points that satisfies the property of local isotropy is not necessarily homogenous: homogeneity is implied by the condition of local isotropy together with assumption of analyticity or regularity. A fractal structure is locally isotropic and thus compatible with the so-called Conditional Cosmological Principle (Mandelbrot 1982), according to which all the points of the system are statistically equivalent.

The only way to study the homogeneity of the large scale structures without a priori assumption (Coleman and Pietronero 1992) is to identify the eventual scale λ_0, if present, beyond which the two point correlation function

$G(r) =< n(r_0)n(r + r_0) >$ shows an evident plateau, within the statistical fluctuations. This means that for $r \geq \lambda_0$ $< n(r_0)n(r + r_0) > \approx < n >^2$ within the fluctuations of order $< n >^{\frac{1}{2}}$, and a definite average density has been reached. Clearly, as the homogeneity implies isotropy $d_{conv} \leq \lambda_0$, and from the knowledge of d_{conv} one cannot infer anything about λ_0.

Several authors (Juszkiewicz et al. 1990, Lahav et al. 1990, and Peacock 1992) have considered the growth of the dipole in the case of different power spectra of fluctuation, under the assumption of small departures from homogeneity. The 'toy-model' described here calls attention in interpreting dipoles if the mean density cannot be defined because in the case of a fractal distribution of matter, as one has density fluctuations extended at all scale, the predictions of the linear theory are not correct. In particular we want to stress that if an average density cannot be defined, in other words if there is not an experimental support for analyticity, under the assumption that light traces mass, no value of the cosmological density parameter Ω_0 (Peebles 1980) can

be inferred from the measurement of the dipole amplitude.

5 Acknowledgements

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6 Figure captions

- Fig.1 The solid line represents

  the Chandrasekhar asymptotic solution ($N \rightarrow \infty$)

  for the distribution $W(|F|)$ of the gravitational force modulus,

  $|F|$, versus $|F|$

  in a random sample with random masses in the three-dimensional euclidean

  space.

  The errors bars refer to

  $W(|F|)$ versus $|F|$

  determined by

  an average of 10 simulations of an homogeneous sample with $< n >$=500.

- Fig.2 Monopole growth with sample depth for the random samples

  of Fig.1. The dependence on depth is linear.

- Fig.3 Dipole modulus growth with sample depth for the random

  sample of Fig.1. The scale of saturation is the distance beyond

  which the distribution isotropy is statistically reached.

- Fig.4 Example of simple deterministic fractal:

  the same structures

  repeat in a self-similar way at different scales, but the

  distribution is non-analytic and non-homogeneous.

The dipole saturates
quickly from every occupied point, and the monopole grows with
the number of points.

- Fig.5 Monopole and dipole modulus behaviour with sample depth for a fractal with D=1.66. One can see that the saturation depth d_{conv} of the dipole modulus is finite, and that the monopole depends on depth as a power law with exponent α.

- Fig.6 Monopole and dipole modulus behaviour with sample depth for a fractal with D=1.85. d_{conv} and α are related to morphological properties of the system and depend from the particular realization of the fractal.

- Fig.7 Monopole and dipole modulus behaviour with sample depth for a fractal with D=1.4, the fractal dimension observed in galaxies and clusters catalogues: the evidence of dipole saturation, together with a monotone growth of the monopole, is not a proof of homogeneity, as it is fully compatible with a non-analytic (fractal) distribution.